

Words and polynomial invariants of the symmetric group in non-commutative variables

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Context

- V k -vector space with basis $\{x_1, x_2, \dots, x_n\}$
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$$T(V) = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots \simeq k\langle x_1, x_2, \dots, x_n \rangle$$
- Invariant algebra of G
$$T(V)^G = k \oplus V^G \oplus (V^{\otimes 2})^G \oplus (V^{\otimes 3})^G \oplus \dots \simeq k\langle x_1, x_2, \dots, x_n \rangle^G$$
- Hilbert-Poincaré series of $T(V)^G$
$$P(T(V)^G) = \sum_{d \geq 0} \dim(V^{\otimes d})^G q^d$$

Theorem (Dick-Formaneck, Kharchenko, Lane)

- i) If $\text{char } k = 0$, then $P(T(V)^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - \text{Tr}(g)q}$.
- ii) As a k -algebra, $T(V)^G$ finitely generated $\iff G$ is scalar.
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Finite combinatorial description of the dimensions and generators of $T(V)^G$ when G is not scalar?

Symmetric group $\mathcal{S}_n$

Action of the symmetric group $\mathcal{S}_n$ on $T(V) \simeq \mathbb{Q}\langle x_1, x_2, \dots, x_n \rangle$

$$\sigma \cdot x_i = x_{\sigma(i)}.$$

$T(V)^{\mathcal{S}_n} \simeq \mathbb{Q}\langle x_1, x_2, \dots, x_n \rangle^{\mathcal{S}_n}$ symmetric polynomials in
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Symmetric group $\mathcal{S}_n$

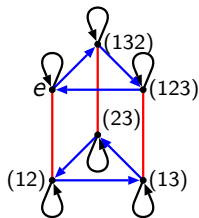
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Combinatorial description for $\dim (V^{\otimes d})^{\mathcal{S}_n}$

For $n = 3$,



$$\begin{aligned} \dim (V^{\otimes 3})^{\mathcal{S}_3} &= \text{number of paths of length 3 from } e \text{ to } e \\ &= |\{bbb, aae, aea, eaa, eee\}| \\ &= 5 \end{aligned}$$

Cayley graph of $\mathcal{S}_n$

Let $S = \{e, (12), (132), \dots, (1\ n \cdots 432)\}$ and e be the identity in $\mathcal{S}_n$.

Cayley graph $\Gamma = \Gamma(\mathcal{S}_n, S)$ of $\mathcal{S}_n$ with generating set S

- vertices are identified with elements of $\mathcal{S}_n$
- for any $g, h \in \mathcal{S}_n$ and $s \in S$,

$$g \bullet \xrightarrow{\text{blue}} \bullet h \quad \text{if } h = gs.$$

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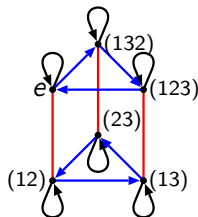
$$g \bullet \xrightarrow{\text{blue}} \bullet h \quad \text{if } h = gs.$$

A word which reduces to g is a path along the edges of Γ from e to g .

A word does not cross e if it has no proper prefix which reduces to e .

Cayley graph of $\mathcal{S}_3$

$\Gamma(\mathcal{S}_3, \{e, (12), (132)\}) :$



$bbbaaba$ reduces to (23) and does cross the identity

$babbaab$ reduces to (23) and does not cross the identity

General Theorem for $\mathcal{S}_n$

Theorem

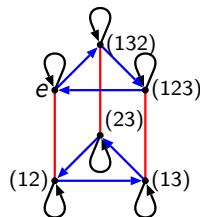
The dimension of $(V^{\otimes d})^{\mathcal{S}_n}$ is equal to the number of words of length d which reduce to e in $\Gamma(\mathcal{S}_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.

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Definitions and notations

- $\lambda \vdash n$ partition of a positive integer n .

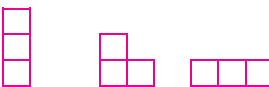
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- **Tableau** of shape $\lambda \vdash n$ is a filling of λ with values in $\{1, 2, \dots, n\}$.
- $S\text{Tab}_n$ is the set of standard tableau with n boxes.

$$S\text{Tab}_3 = \left\{ \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \right\}$$

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- V^λ irreducible S_n -module with character χ^λ .

Proof uses Solomon's descent algebra of S_n

$$I = \{1, 2, \dots, n-1\}$$

$$Des(\sigma) = \{i \in I \mid \sigma(i) > \sigma(i+1)\}$$

$$\sigma \leftrightarrow (P(\sigma), Q(\sigma))$$

descent set of $\sigma \in S_n$

Robinson-Schensted correspondence

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$$\bullet \mathcal{Q}_n = \mathcal{L}\{z_t \mid t \in STab_n\}$$

Proof uses Solomon's descent algebra of S_n

There is an algebra morphism [Solomon]

$$\theta : \Sigma(\mathcal{S}_n) \rightarrow \mathbb{Z}\text{Irr}(\mathcal{S}_n)$$

and a linear map [Poirier-Reutenauer]

$$\begin{array}{ccc} \tilde{\theta} : \mathcal{Q}_n & \longrightarrow & \mathbb{Z}\text{Irr}(\mathcal{S}_n) \\ z_t & \longmapsto & \chi^{\text{shape}(t)} \end{array}$$

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$$V = V^{(n)} \oplus V^{(n-1,1)}$$

$$\tilde{\theta}(z_{\boxed{1\,2\,3\,\cdots\,n}} + z_{\boxed{2\,1\,3\,4\,\cdots\,n}}) = \theta(e + d_{\{1\}}) = \chi^{(n)} + \chi^{(n-1,1)} = \chi^V$$

Idea of the proof

Proposition

The dimension of $(V^{\otimes d})^{S_n}$ is equal to the coefficient of e in $(e + d_{\{1\}})^d$.

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The dimension of $(V^{\otimes d})^{S_n}$ is equal to the coefficient of e in $(e + d_{\{1\}})^d$.

$$\dim (V^{\otimes 3})^{S_3} = \text{multiplicity of the trivial in } V^{\otimes 3}$$

$$\begin{aligned}(e + d_{\{1\}})^3 &= 5e + 5d_{\{2\}} + 4d_{\{1\}} + 4d_{\{1,2\}} \\ &= 5z_{\begin{smallmatrix} \boxed{1123} \end{smallmatrix}} + 5z_{\begin{smallmatrix} \boxed{3} \\ \boxed{12} \end{smallmatrix}} + 4z_{\begin{smallmatrix} \boxed{2} \\ \boxed{13} \end{smallmatrix}} + 4z_{\begin{smallmatrix} \boxed{3} \\ \boxed{2} \\ \boxed{1} \end{smallmatrix}}\end{aligned}$$

$$\downarrow \tilde{\theta}$$

$$(\chi^V)^3 = 5\chi^{(3)} + 5\chi^{(2,1)} + 4\chi^{(2,1)} + 4\chi^{(1,1,1)}$$

Lemma

The coefficient of e in $(e + d_{\{1\}})^d$ is equal to the number of words of length d which reduce to e in $\Gamma(\mathcal{S}_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.

Idea of the proof

Lemma

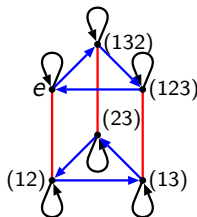
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Set $a = (12)$ and $b = (132)$.

$$(e + d_{\{1\}})^3 = (e + a + b)^3 = 5e + 5(12) + 4(23) + 4(123) + 5(132) + 4(13)$$

$$\Gamma(\mathcal{S}_3, \{e, (12), (132)\})$$

bbb
 aae
 aea
 eea
 eee



Free generators for $T(V)^{\mathcal{S}_n}$

Proposition

The number of free generators of $T(V)^{\mathcal{S}_n}$ as an algebra are counted by the words which reduce to the identity without crossing the identity in $\Gamma(\mathcal{S}_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.

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The free generators of $T(V)^{S_3}$ are counted by

e	aa	bbb	abab	abbbba	beaba	...
	aea	baba	baabb	baeba		
		bebb	bbaab	babea		
		bbab	aeab	beebb		
		aeaa	abeab	bebeb		
			abae	bbeeb		
				aeaa		
1	1	2	5	13	...	

the odd indexed Fibonacci numbers.

Multiplicity of V^λ in $V^{\otimes d}$

$$\{V^\lambda\}_{\lambda \vdash n}$$

irreducible $\mathcal{S}_n$ -modules

$$\sigma \leftrightarrow (P(\sigma), Q(\sigma))$$

Robinson-Schensted correspondence

Theorem

Let $\lambda \vdash n$. The multiplicity of V^λ in $V^{\otimes d}$ is equal to

$$\sum_{\substack{t \in STab_n \\ \text{shape}(t) = \lambda}} |w(\sigma_t, d; \Gamma)|,$$

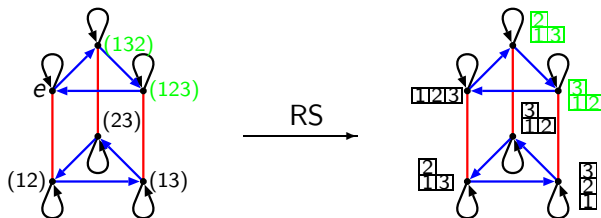
where

- $\Gamma = \Gamma(\mathcal{S}_n, \{e, (12), (132), \dots, (1\ n \dots 432)\})$
- $\sigma_t \in \mathcal{S}_n$ is such that $Q(\sigma_t) = t$
- $w(\sigma_t, d; \Gamma)$ is the set of words of length d which reduce to σ_t in Γ .

In particular, the multiplicity of the trivial is $|w(e, d; \Gamma)|$.

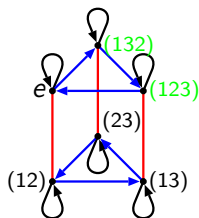
Decomposition of the $\mathcal{S}_3$ -module $V^{\otimes 3}$

Consider $\Gamma = \Gamma(\mathcal{S}_3, \{e, (12), (132)\})$ and set $a = (12)$ and $b = (132)$

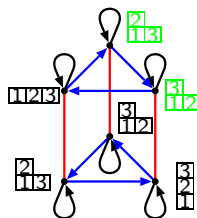


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RS

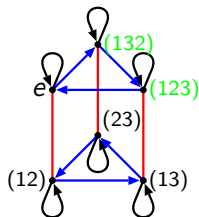


$$|w(\sigma_{\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{smallmatrix}}, 3; \Gamma)|$$

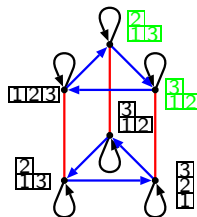
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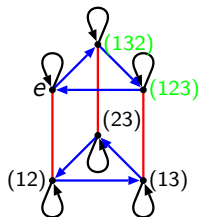
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$$|w(\sigma_{\begin{smallmatrix} 3 \\ 1 & 1 & 2 \end{smallmatrix}}, 3; \Gamma)| + |w(\sigma_{\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}}, 3; \Gamma)| = |\{aba, bbe, beb, ebb\}|$$

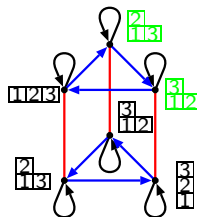
$$+ |\{aab, baa, bee, ebe, eeb\}| = 9$$

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RS $\longrightarrow$



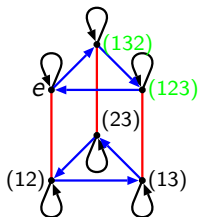
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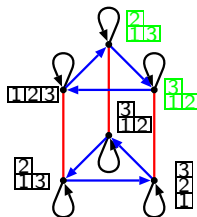
$$|w(\sigma_{\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}}, 3; \Gamma)| = |\{bba, abe, aeb, eab\}| = 4$$

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$$|w(\sigma_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}, 3; \Gamma)| = |\{bbb, aae, aea, eaa, eee\}| = 5$$

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$$V^{\otimes 3} = 5 V^{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}} \oplus 9 V^{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \oplus 4 V^{\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix}}$$

Applications: set partitions

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$$B \circ C = \begin{cases} \{B_1 \cup (C_1 + d), \dots, B_k \cup (C_k + d), (C_{k+1} + d), \dots, (C_\ell + d)\} & \text{if } k \leq \ell \\ \{B_1 \cup (C_1 + d), \dots, B_\ell \cup (C_\ell + d), B_{\ell+1}, \dots, B_k\} & \text{if } k > \ell. \end{cases}$$

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The set partitions of $\{1, 2, 3\}$ are

1|2|3 non-splitable

1|23 non-splitable

12|3 = $1 \circ 1|2$

13|2 = $1|2 \circ 1$

123 = $1 \circ 1 \circ 1$

Symmetric polynomials in non-commutative variables

X_n set of variables $x_1, x_2, \dots, x_n$

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[Wolf, Rosas and Sagan]

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- $\mathbb{Q}\langle X_n \rangle^{S_n} = \mathcal{L}\{\mathbf{m}_A(X_n) \mid A \text{ set partition with at most } n \text{ parts}\}$

Symmetric polynomials in non-commutative variables

X_n set of variables $x_1, x_2, \dots, x_n$

$T(V)^{S_n} \simeq \mathbb{Q}\langle X_n \rangle^{S_n}$ symmetric polynomials in non-commutative variables
[Wolf, Rosas and Sagan]

- $\mathbb{Q}\langle X_n \rangle^{S_n} = \mathcal{L}\{\mathbf{m}_A(X_n) \mid A \text{ set partition with at most } n \text{ parts}\}$
- $\mathbb{Q}\langle X_n \rangle^{S_n}$ freely generated by
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- $$P(\mathbb{Q}\langle X_n \rangle^{S_n}) = \sum_{k=0}^n \frac{q^k}{(1-q)(1-2q)\cdots(1-kq)}$$

number of set partitions
with k parts [Comtet]

Symmetric polynomials in non-commutative variables

$\dim \mathbb{Q}\langle X_n \rangle_d^{S_n} = \text{number of set partitions of } [d] \text{ into at most } n \text{ parts}$

Symmetric polynomials in non-commutative variables

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Basis for $\mathbb{Q}\langle x_1, x_2, x_3 \rangle_3^{S_3}$:

- $\mathbf{m}_{1|2|3} = x_1x_2x_3 + x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2 + x_3x_2x_1,$
- $\mathbf{m}_{12|3} = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2,$
- $\mathbf{m}_{13|2} = x_1x_2x_1 + x_1x_3x_1 + x_2x_1x_2 + x_2x_3x_2 + x_3x_1x_3 + x_3x_2x_3,$
- $\mathbf{m}_{1|23} = x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2,$
- $\mathbf{m}_{123} = x_1^3 + x_2^3 + x_3^3.$

Symmetric polynomials in non-commutative variables

$$\dim \mathbb{Q}\langle X_n \rangle_d^{\mathcal{S}_n} = \text{number of set partitions of } [d] \text{ into at most } n \text{ parts}$$

Basis for $\mathbb{Q}\langle x_1, x_2, x_3 \rangle_3^{\mathcal{S}_3}$:

- $\mathbf{m}_{1|2|3} = x_1x_2x_3 + x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2 + x_3x_2x_1,$
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- $\mathbf{m}_{1|23} = x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2,$
- $\mathbf{m}_{123} = x_1^3 + x_2^3 + x_3^3.$

Words of length 3 which reduce to the identity in $\Gamma(\mathcal{S}_3, \{e, (12), (132)\})$:

$$\{bbb, aae, aea, eaa, eee\}.$$

Symmetric polynomials in non-commutative variables

number of free generators
for $\mathbb{Q}\langle X_n \rangle_d^{S_n}$ = number of non-splitable set partitions
of $[d]$ into at most n parts

Symmetric polynomials in non-commutative variables

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Free generators for $\mathbb{Q}\langle x_1, x_2, x_3 \rangle_4^{S_3}$:

- $\mathbf{m}_{1|23|4} = x_1 x_2^2 x_3 + x_1 x_3^2 x_2 + x_2 x_1^2 x_3 + x_2 x_3^2 x_1 + x_3 x_1^2 x_2 + x_3 x_2^2 x_1,$
- $\mathbf{m}_{13|2|4} = x_1 x_2 x_1 x_3 + x_1 x_3 x_1 x_2 + x_2 x_1 x_2 x_3 + x_2 x_3 x_2 x_1 + x_3 x_1 x_3 x_2 + x_3 x_2 x_3 x_1,$
- $\mathbf{m}_{1|24|3} = x_1 x_2 x_3 x_2 + x_1 x_3 x_2 x_3 + x_2 x_1 x_3 x_1 + x_2 x_3 x_1 x_3 + x_3 x_1 x_2 x_1 + x_3 x_2 x_1 x_2,$
- $\mathbf{m}_{1|2|34} = x_1 x_2 x_3^2 + x_1 x_3 x_2^2 + x_2 x_1 x_3^2 + x_2 x_3 x_1^2 + x_3 x_1 x_2^2 + x_3 x_2 x_1^2,$
- $\mathbf{m}_{1|234} = x_1 x_2^3 + x_1 x_3^3 + x_2 x_1^3 + x_2 x_3^3 + x_3 x_1^3 + x_3 x_2^3.$

Symmetric polynomials in non-commutative variables

$$\text{number of free generators for } \mathbb{Q}\langle X_n \rangle_d^{S_n} = \text{number of non-splitable set partitions of } [d] \text{ into at most } n \text{ parts}$$

Free generators for $\mathbb{Q}\langle x_1, x_2, x_3 \rangle_4^{S_3}$:

- $\mathbf{m}_{1|23|4} = x_1x_2^2x_3 + x_1x_3^2x_2 + x_2x_1^2x_3 + x_2x_3^2x_1 + x_3x_1^2x_2 + x_3x_2^2x_1,$
- $\mathbf{m}_{13|2|4} = x_1x_2x_1x_3 + x_1x_3x_1x_2 + x_2x_1x_2x_3 + x_2x_3x_2x_1 + x_3x_1x_3x_2 + x_3x_2x_3x_1,$
- $\mathbf{m}_{1|24|3} = x_1x_2x_3x_2 + x_1x_3x_2x_3 + x_2x_1x_3x_1 + x_2x_3x_1x_3 + x_3x_1x_2x_1 + x_3x_2x_1x_2,$
- $\mathbf{m}_{1|2|34} = x_1x_2x_3^2 + x_1x_3x_2^2 + x_2x_1x_3^2 + x_2x_3x_1^2 + x_3x_1x_2^2 + x_3x_2x_1^2,$
- $\mathbf{m}_{1|234} = x_1x_2^3 + x_1x_3^3 + x_2x_1^3 + x_2x_3^3 + x_3x_1^3 + x_3x_2^3.$

Words of length 4 which reduce to the identity without crossing the identity in $\Gamma(S_3, \{e, (12), (132)\})$:

$$\{abab, baba, bebb, bbeb, aeea\}.$$

Corollary

The number of set partitions of $[d]$ into at most n parts equals the number of words of length d which reduce to the identity in $\Gamma(\mathcal{S}_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.

Set partitions and words

Corollary

The number of set partitions of $[d]$ into at most n parts equals the number of words of length d which reduce to the identity in $\Gamma(S_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.

Corollary

The number of non-splitable set partitions of $[d]$ into at most n parts equals the number of words of length d which reduce to the identity without crossing it in $\Gamma(S_n, \{e, (12), (132), \dots, (1\ n \cdots 432)\})$.